Group velocity and characteristic wave curves of Lamb waves in composites: Modeling and experiments

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Abstract

The propagation characteristics of Lamb waves in composites, with emphasis on group velocity and characteristic wave curves, are investigated theoretically and experimentally. In particular, the experimental study focuses on the existence of multiple higher-order Lamb wave modes that can be observed from piezoelectric sensors by the excitation of ultrasonic frequencies. Using three-dimensional (3-D) elasticity theory, the exact dispersion relations governed by transcendental equations are numerically solved for an infinite number of possible wave modes. For symmetric laminates, a robust method by imposing boundary conditions on the mid-plane and top surface is proposed to separate symmetric and anti-symmetric wave modes. A new semi-exact method is developed to calculate group velocities of Lamb waves in composites. Meanwhile, three characteristic wave curves: velocity, slowness, and wave curves are adopted to analyze the angular dependency of Lamb wave propagation. The dispersive and anisotropic behavior of Lamb waves in a two different types of symmetric laminates is studied in detail theoretically. In the experimental study, two surface-mounted piezoelectric actuators are excited either symmetric or anti-symmetric wave modes with narrowband signals, and a Gabor wavelet transform is used to extract group velocities from arrival times of Lamb wave received by a piezoelectric sensor. In comparison with the results from the theory and experiment, it is confirmed that multiple higher-order Lamb waves can be excited from piezoelectric actuators and the measured group velocities agree well with those from 3-D elasticity theory.

Keywords: Lamb waves; Dispersion relation; Phase and Group velocity; Slowness; Structural health monitoring

1. Introduction

The integrity and degradation of composite structures have been traditionally evaluated by nondestructive evaluation (NDE) [1–3], now being potentially assessed by structural health monitoring (SHM) [4], to assure the performance of these structures. For the active diagnosis that utilizes ultrasonic transient waves for damage detection, localization and then assessment of the damage, understanding the wave propagation characteristics in composites is essential for successful application of these techniques. The wave propagation in composites is complex due to the nature of heterogeneity of the constituents, inherent material anisotropy, and the multi-layered construction, which leads to the velocity of wave mode being macroscopically dependent on the laminate layup, the direction of wave propagation, frequency, and interface conditions.

When elastic waves propagate in isotropic plate-like structures, they would experience repeated reflections at the top and bottom surfaces alternately and the resulting wave propagation from their mutual interference is guided by the plate surfaces. The guided wave can be modeled by imposing surface boundary conditions on the equations of motion. However, this approach introduces the dispersion phenomenon; that is, the velocity of propagation of the guided wave along the plate being a function of frequency,
or equivalently, wavelength. The dispersion relations for an elastic isotropic plate with infinite extent in plane strain state were first derived by Lamb [5]. With generalization the guided waves propagating along the plane of an elastic plate with traction-free boundaries are usually called Lamb waves. Since guided waves remain confined inside the structure, they can travel over long distances enabling the inspection of a large area with only a limited number of sensors. This property makes them well suited in SHM to the ultrasonic inspection of plate-like aircraft components, missile cases, pressure vessel, oil tanks, pipelines, etc. In isotropic plates, the guided waves can be classified into three types according to their polarizations (the direction of the displacement vector). Those polarized in the plane perpendicular to the plate, say in the $x-z$ plane shown in Fig. 1, are called symmetric (or extensional, $S$) waves and anti-symmetric (or flexural, $A$) waves, and those polarized in the plane of the plate (in the $y$-axis) are called shear horizontal ($SH$) waves. The $SH$ waves can also be either symmetric or anti-symmetric with respect to the mid-plane. The $S$ and $A$ waves are governed by the plane strain state (the displacements $u$ and $w$); while $SH$ waves by antiplane strain (displacement $v$ only). Conventionally, $S_n$ and $A_n$ with subscript $n = 0, 1, 2, 3 \ldots$ represent symmetric and anti-symmetric Lamb wave modes, respectively; $SH_n$ with even and odd subscript $n$ denotes symmetric and anti-symmetric $SH$ waves, respectively.

For waves propagating in multi-layered composites, the wave interactions depend upon the constituent properties, geometry, direction of propagation, frequency, and interfacial conditions. If the wavelengths are significantly longer than the sizes of the constituents of composites (fiber diameters and spacing), each lamina can be treated as an equivalent homogeneous orthotropic or transversely isotropic material with the symmetry axis parallel to the fibers. Tauchert and Guzelsu [6] measured scattering in boron/epoxy material with the symmetry axis parallel to the fibers. Tauc- Alertant homogeneous orthotropic or transversely isotropic parameters and spacing), each lamina can be treated as an equivalent homogeneous orthotropic or transversely isotropic material with the symmetry axis parallel to the fibers. Tauchert and Guzelsu [6] measured scattering in boron/epoxy material with the symmetry axis parallel to the fibers. Tauc-

![Diagram of Lamb waves propagating in a composite laminate.](image)

In general, there are two theoretical approaches to investigate Lamb waves in composites: one is exact solutions by 3-D elasticity theory, and the other is approximate solutions by plate theories. For the approach using 3-D elasticity, Nayfeh and Chimenti [7] gave dispersion relations of Lamb waves in a composite lamina. Later, Nayfeh [8] developed a transfer matrix technique to obtain the dispersion curves in laminates. Yuan and Hsieh [9] obtained the exact solutions of dispersion of Lamb waves in composite shells and compared them with Flügge shell theory. However, all these studies above only obtained the dispersion relations of phase velocity, not group velocity. Recently Neau [10] obtained both group velocity dispersions and wave curves in a lamina, and also compared with experiments. Chimenti [3] presented a comprehensive review on theories and applications of Lamb waves. Although the exact solutions provide accurate results, the computation for dispersion characteristics of multi-layered composites is intensive because of the transcendental equations, not to mention the transient wave response of composites. To make the solutions tractable, many researchers have strived to approximate solutions by laminated plate theories [11–13]. However these studies did not obtain the group velocity dispersions of Lamb waves. Liu and Xi [14] deduced approximate solutions of group velocities and characteristic wave curves by assuming quadratic interpolation functions through the thickness direction. Since most of Lamb wave-based damage detection techniques evaluate arrival time (or time-of-flight) of scattering waves from damage, then by knowing the group velocity, the location of damage can be determined [15,16].
Experimental studies on the determination of dispersive curves have been focused mainly on isotropic plates. The most effective setup to excite and receive multiple modes of Lamb waves is the laser source together with interferometer [17] because of their superior sensitivity and wide band nature. Moreover the phase velocities of Lamb waves are normally measured by wedge transducer [18,19]. For SHM using guided waves the frequencies used are typically much lower than in NDE methods, typically above few MHz. If piezoelectric actuators/sensors are used, they are often operated in a frequency up to 1 MHz. In addition, the input signals often used are narrow band (finite tone bursts or wave packets) to prevent the excitation of higher frequency modes. For fiber reinforced composites, another complexity of wave attenuation can be caused by the viscoelastic nature of the resin and by scattering from the fibers and other heterogeneities in the material at higher frequencies.

This paper is organized as follows. In the next Section, the exact dispersion relations of symmetric and anti-symmetric wave modes in a lamina are formulated from 3-D elasticity. Then the formulation is extended to a composite laminate with an arbitrary stacking sequence. For symmetric laminates, a robust method by imposing boundary conditions on both mid-plane and top surface is developed to separate the symmetric and anti-symmetric modes. Section 3 discusses the physical relations between phase velocity, slowness, and group velocity of Lamb waves; and the characteristic wave curves are introduced to analyze the anisotropic and dispersion of Lamb waves. Meanwhile a new semi-exact method is developed to compute wave curves by performing finite difference on exact solutions of two adjacent slowness curves. Section 4 implements the methods to show the dispersions and characteristic wave curves of Lamb waves in two types of laminates. Then Section 5 investigates the experiment procedure of exciting/receiving Lamb waves in laminates, applies Gabor wavelet transform to extract the group velocities arrival times of all wave modes, and compares experimental results with theoretical prediction. Finally, in Section 6 some conclusions and guidelines for the practice of NDE and SHM are drawn.

2. Formulation of Lamb waves in composites by 3-D elasticity theory

Transient waves traveling in the composites along arbitrary direction in general cause disturbance involving all three displacement components, so called generalized plane deformation due to anisotropy of the material [20]. In this Section, a theoretical framework is developed first for displacement expression of plane harmonic waves using 3-D elasticity. In Section 2.1, focus is on the Lamb waves in a single lamina (monoclinic plate) where a compact closed-form dispersion relation can be derived by separating symmetric and anti-symmetrical modes using trigonometric functions through the lamina thickness. Special cases where the waves propagate in the symmetry axis of the material such that the uncoupling of S and A waves and SH waves are considered. Then a modified exponential form in the thickness direction is proposed for deriving the dispersion relation for a composite laminate, with special emphasis on the symmetric laminates.

A Cartesian coordinate system is used with z-axis normal to the mid-plane of a composite laminate spanned by x and y axes. Two outer surfaces of the laminate are at z = ±h/2. A packet of Lamb waves propagates in an arbitrary direction θ, which is defined counterclockwise relative to the x-axis. Each layer of the composite laminate with an arbitrary orientation in the global coordinate system (x,y,z) is considered as a monoclinic material having x–y as a plane of symmetry, the stress–strain relations therefore take the following matrix form:

\[
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xz} \\
\tau_{xy} \\
\tau_{yz}
\end{pmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\
C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\
0 & 0 & 0 & C_{44} & C_{45} & 0 \\
0 & 0 & 0 & C_{45} & C_{55} & 0 \\
C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}
\end{bmatrix}
\begin{pmatrix}
ev_x \\
ev_y \\
ev_z \\
g_{je} \\
g_{je} \\
g_{je}
\end{pmatrix}
\]

When the global coordinate system (x,y,z) does not coincide with the principal material coordinate system (x′,y′,z) of each layer but makes an angle φ with the x-axis shown in Fig. 1, the stiffness matrix \(C_{ij}(i,j = 1,2,3,\ldots,6)\) in (x,y,z) system can be obtained from the lamina stiffness matrix \(C_{ij}'\) in (x′,y′,z) system by using a transformation matrix method [21]. The lamina is orthotropic or transversely isotropic with respect to the principal material axes in (x′,y′,z) and its lamina stiffness matrix \(C_{ij}'\) can be calculated from the lamina engineering material properties \(E_{ik} \), \(G_{kh} \) and \(G_{kl}(k,l = 1,2,3)\) [22].

The linear engineering strain–displacement relations are

\[
e_{x} = u_{x}, \quad e_{y} = v_{y}, \quad e_{z} = w_{z}, \quad \gamma_{je} = v_{x} + w_{y}, \quad \gamma_{je}
\]

where subscript comma indicates partial differential; \(u, v, \) and \(w\) are the displacements in the \(x, y, \) and \(z\)-directions, respectively.

The equations of motion in the absence of body forces are governed by:

\[
\sigma_{xx} + \tau_{xy} + \tau_{xz} = \rho \ddot{u} \quad (3a) \\
\tau_{yx} + \sigma_{yy} + \tau_{yz} = \rho \ddot{v} \quad (3b) \\
\tau_{zx} + \tau_{zy} + \sigma_{zz} = \rho \ddot{w} \quad (3c)
\]

where \(\rho\) is the mass density of the lamina, and dot denotes time derivative. The traction-free boundary conditions on the top and bottom surfaces are

\[
\sigma_{z} = \tau_{xz} = \tau_{yz} = 0, \text{ at } z = \pm h/2 \quad (4)
\]

Since the Lamb waves travel along the plane of a plate with traction-free boundaries but are standing waves in the \(z\)-direction of the plate, the wave motion may be
expressed by superposition of plane harmonic waves. Each plane harmonic wave traveling in the direction of wave normal $k$ is represented by

$$\{u, v, w\} = \{U(z), V(z), W(z)\}e^{i(k_x x + k_y y) - \omega t}$$

(5)

where $k = [k_x, k_y]^T$ and its magnitude is $k = |k| = \sqrt{k_x^2 + k_y^2} = \omega/c_p$ is the wave number. $\lambda = 2\pi/\omega$ is the wavelength. $\omega$ is the angular frequency and $c_p$ is the phase velocity. Note that $k$ points the direction of propagation. In the $x$-$y$ plane, $k = [\cos \theta, \sin \theta]^T$ where $\theta$ is the direction of wave propagation.

Substituting Eqs. (5) and (2) in (1) gives the expressions of stresses in each layer: substituting Eqs. (5) and (2) in (1) gives the expressions of stresses in each layer:

$$\sigma_x = [C_{11}k_x U + C_{12}k_y V - iC_{13}W'] + C_{16}(k_x U + k_y V)]e^{i(k_x x + k_y y) - \omega t}$$

(6a)

$$\sigma_y = [C_{12}k_x U + C_{22}k_y V - iC_{23}W'] + C_{16}(k_x U + k_y V)]e^{i(k_x x + k_y y) - \omega t}$$

(6b)

$$\sigma_z = [C_{13}k_x U + C_{23}k_y V - iC_{33}W'] + C_{16}(k_x U + k_y V)]e^{i(k_x x + k_y y) - \omega t}$$

(6c)

$$\tau_{xz} = [C_{44}(V' + ik_k)W] + C_{45}(U' + ik_k)W)e^{i(k_x x + k_y y) - \omega t}$$

(6d)

$$\tau_{yz} = [C_{45}(U' + ik_k)W] + C_{43}(V' + ik_k)W)e^{i(k_x x + k_y y) - \omega t}$$

(6e)

$$\tau_{xy} = [C_{66}(k_x U + k_y V)] + C_{66}(k_x U + k_y V)]e^{i(k_x x + k_y y) - \omega t}$$

(6f)

where the prime denotes the derivative with respect to $z$.

Substituting Eq. (6) into Eq. (3), the equations of motion of each layer become

$$-C_{55}U'' - C_{45}V^{'''} + (C_{11}k_x^2 + 2C_{16}k_x k_y + C_{16}k_y^2 - \rho \omega^2)U$$

$$+ (C_{16}k_x^2 + C_{12}k_x k_y + C_{66}k_y^2 + C_{66}k_y^2 - \rho \omega^2)U$$

$$+ i((C_{13} + C_{55})k_x + (C_{36} + C_{45})k_y)W' = 0$$

(7a)

$$-C_{45}U'' - C_{44}V^{'''} + (C_{16}k_x^2 + C_{12}k_x k_y + C_{66}k_y^2 + C_{66}k_y^2 - \rho \omega^2)U$$

$$+ (C_{16}k_x^2 + 2C_{26}k_x k_y + C_{26}k_y^2 - \rho \omega^2)U$$

$$- i((C_{36} + C_{45})k_x + (C_{23} + C_{44})k_y)W' = 0$$

(7b)

$$-i((C_{13} + C_{55})k_x + (C_{36} + C_{45})k_y)U' - i((C_{36} + C_{45})k_x + (C_{23} + C_{44})k_y)W'$$

$$+ C_{44}k_x^2 + C_{44}k_y^2 - \rho \omega^2)W' = 0$$

(7c)

### 2.1. Lamb waves in a composite lamina

In an off-axis lamina, the solutions of Eq. (7) can be simply separated into symmetric and anti-symmetric wave modes, which render the analytical representation particularly simple:

$$U_s = A_s \cos \xi z, \quad V_s = B_s \cos \xi z, \quad W_s = C_s \sin \xi z$$

$$U_a = A_a \sin \xi z, \quad V_a = B_a \sin \xi z, \quad W_a = C_a \cos \xi z$$

(8a)

(8b)

where $\xi$ is an unknown variable to be determined later; moreover the subscripts s and a represent symmetric and anti-symmetric modes, respectively.

First symmetric modes are considered. Substitution of Eq. (8a) into equations of motion, Eq. (7), leads to an expression in a matrix form

$$\begin{pmatrix}
\Gamma_{11} - \rho \omega^2 & \Gamma_{12} - \rho \omega^2 & \Gamma_{13} - \rho \omega^2 \\
\Gamma_{12} - \rho \omega^2 & \Gamma_{22} - \rho \omega^2 & \Gamma_{23} - \rho \omega^2 \\
\Gamma_{13} - \rho \omega^2 & \Gamma_{23} - \rho \omega^2 & \Gamma_{33} - \rho \omega^2
\end{pmatrix}
\begin{pmatrix}
A_s \\
B_s \\
C_s
\end{pmatrix} = 0$$

(9)

where the bar indicates complex conjugate. The elements in the above matrix defined by $(\mathbf{I} - \rho \omega^2 \mathbf{I})$ are listed as follows:

$$\Gamma_{11} = C_{11}k_x^2 + 2C_{16}k_x k_y + C_{66}k_y^2 + C_{55} \xi^2$$

(10a)

$$\Gamma_{12} = C_{16}k_x^2 + (C_{12} + C_{66})k_x k_y + C_{26}k_y^2 + C_{45} \xi^2$$

(10b)

$$\Gamma_{13} = -i[(C_{13} + C_{55})k_x + (C_{36} + C_{45})k_y] \xi$$

(10c)

$$\Gamma_{22} = C_{66}k_x^2 + 2C_{26}k_x k_y + C_{22}k_y^2 + C_{44} \xi^2$$

(10d)

$$\Gamma_{23} = -i[(C_{36} + C_{45})k_x + (C_{23} + C_{44})k_y] \xi$$

(10e)

$$\Gamma_{33} = C_{55}k_x^2 + 2C_{44}k_x k_y + C_{44}k_y^2 + C_{33} \xi^2$$

(10f)

where $\mathbf{I}$ is a $3 \times 3$ identity matrix.

Eq. (9) is a standard linear eigenvalue problem of a Hermitian matrix $\Gamma$. If the matrix is positive definite, it can be shown that the eigenvalues $\rho \omega^2$ of $\Gamma$ are positive and non-zero, furthermore the right eigenvectors follow the orthogonality relation [21].

Following the same procedure for the anti-symmetric mode, the resulting matrix becomes $(\mathbf{I} - \rho \omega^2 \mathbf{I})$. If Hermitian matrix $\Gamma$ is positive definite and so is $\overline{\Gamma}$, it can be readily proved that the eigenvalues of the symmetric mode are also the eigenvalues of the anti-symmetric mode.

For nontrivial solutions of $A_s$, $B_s$, and $C_s$ in Eq. (9), vanishing the determinant of the $3 \times 3$ matrix $(\mathbf{I} - \rho \omega^2 \mathbf{I})$ yields the following sixth-order polynomial in $\xi$:

$$\xi^6 + \xi_1 \xi^4 + \xi_2 \xi^2 + \xi_3 = 0$$

(11)

where $\xi_j (j = 1, 2, 3)$ are real-valued coefficients of $C_{ij}$, $k$, and $\rho \omega^2$. If variable $\xi$ is redefined as $k \xi$, the coefficients $\xi_j$ are functions of $C_{ij}$, $\theta$, and $\rho \omega^2$. Roots of this equation can be obtained explicitly from known formulas since the equation can be reduced to a cubic polynomial in terms of $\xi^2$.

In general there are three positive, nonzero, and discrete $\xi_j (j = 1, 2, 3)$. For each $\xi_j$ in symmetric modes, $B_s$ and $C_s$ related to symmetric modes can be expressed in terms of $A_s$ via Eq. (9) as

$$B_s = \frac{\Gamma_{11} - \rho \omega^2 \Gamma_{23} - \Gamma_{13} \Gamma_{13} A_s \equiv RA_s}{\Gamma_{13}(\Gamma_{22} - \rho \omega^2) - \Gamma_{12} \Gamma_{23}}$$

(12a)

$$C_s = \frac{\Gamma_{12} - \rho \omega^2 \Gamma_{23} - \Gamma_{13} \Gamma_{23} A_s \equiv CS_a}{\Gamma_{13}(\Gamma_{22} - \rho \omega^2) - \Gamma_{12} \Gamma_{23}}$$

(12b)

and similarly for anti-symmetric modes, $B_s = RA_s$ and $C_s = -iSA_s$. With the above equations, the polarization displacement vectors are determined from the three roots. Consequently, the general solution of Eq. (8) is
\[ \left\{ U_x, V_x, W_x \right\} = \sum_{j=1}^{3} A_j \left\{ \cos \xi_x, R_j \cos \xi_x, i S_j \sin \xi_x \right\} \]
\[ \left\{ U_y, V_y, W_y \right\} = \sum_{j=1}^{3} A_j \left\{ \sin \xi_y, R_j \sin \xi_y, -i S_j \cos \xi_y \right\} \]

Substituting of Eq. (13) in Eq. (6) and considering Eq. (4), \( \sigma_z, \tau_{yz}, \) and \( \tau_{xz} \) are rearranged as
\[ \left( \sigma_z, \tau_{yz}, \tau_{xz} \right)_{\text{sym}} = \sum_{j=1}^{3} \left[ H_{1j} \sin(\xi_y + \varphi), H_{3j} \cos(\xi_y + \varphi), H_{3j} \cos(\xi_z + \varphi) \right] A_j = 0 \] (14)
where \( \varphi = 0 \) and \( \pi/2 \) represent anti-symmetric and symmetric Lamb wave modes, respectively; and
\[
\begin{align*}
H_{1j} &= C_{13} x + C_{23} k_x R_j + C_{33} S_j + C_{36} (k_y + k_x R_j) \quad (15a) \\
H_{3j} &= C_{44} (\xi_y R_j + k_y S_j) + C_{45} (\xi_z + k_x S_j) \quad (15b) \\
H_{3j} &= C_{45} (\xi_y R_j + k_y S_j) + C_{55} (\xi_z + k_x S_j) \quad (15c)
\end{align*}
\]
The existence of a nontrivial solution of Eq. (14) leads to closed-form dispersion relations as
\[
\begin{align*}
&H_{11}(H_{22} H_{33} - H_{23} H_{32}) \tan(\xi_j h/2 + \varphi) + H_{12}(H_{23} H_{31} - H_{21} H_{33}) \tan(\xi_j h/2 + \varphi) + H_{13}(H_{21} H_{32} - H_{22} H_{31}) \tan(\xi_j h/2 + \varphi) = 0 \quad (16)
\end{align*}
\]
Eq. (16) is a transcendental equation implicitly relating \( \omega \) to \( k \). For a fixed \( \theta \), a numerical iterative root-finding method is employed to compute the admissible \( \omega \) for a range of \( k \)-values, leading to dispersion relations of Lamb wave modes in the direction of propagation. Furthermore in general the frequency \( \omega \) of each mode is single-valued function of \( k \).

2.2. Lamb waves in a composite laminate

In formulating Lamb waves in a laminate, the interfaces between layers are assumed to be perfectly bonded. The displacement components of each layer in the \( z \)-axis Eq. (8) needs to be modified in exponential forms to accommodate the inhomogeneity of the multi-layered laminates.
\[
\begin{align*}
U &= Ae^{iz}, & V &= Be^{iz}, & W &= -iCe^{iz} 
\end{align*}
\]
Substituting these expressions into the equations of motion, Eq. (7) may be rearranged in a matrix form
\[ (\mathbf{G} - \rho \omega^2 \mathbf{I}) \left\{ \begin{array}{c} A \\ B \\ C \end{array} \right\} = 0 \] (18)
The nontrivial solution for \( A, B, \) and \( C \) yields the sixth-order polynomial in terms of \( \xi \) shown in Eq. (11). The six roots for \( \xi \) can be arranged in three pairs as \( \xi_{j+1} = -\xi_j, \) \( j = 1, 3, 5 \). For each \( \xi_j \), \( B \) and \( C \) can be expressed in terms of \( A \) via Eq. (18) as \( B_i = R_i A_i \) and \( C_i = -S_i A_i \) \( (i = 1, 2, 3, \ldots, 6) \). Further, \( R_{j+1} = -R_j \) and \( S_{j+1} = -S_j \) \( (j = 1, 3, 5) \).
Consequently, the general solution of Eq. (17) in each lamina is
\[ \left\{ U, V, W \right\} = e^{i(k_x x + k_y y - \omega t)} \sum_{j=1}^{6} A_j \left\{ 1, R_j, S_j \right\} e^{i\xi_j z} \] (19)
The interlaminar stress components, \( \sigma_z, \tau_{yz}, \) and \( \tau_{xz} \), in each lamina may be expressed as
\[ \left\{ \sigma_z, \tau_{yz}, \tau_{xz} \right\} = ike^{i(k_x x + k_y y - \omega t)} \sum_{j=1}^{6} A_j \left\{ H_{1j}, H_{2j}, H_{3j} \right\} e^{i\xi_j z} \] (20)
and \( H_{1(j+1)} = H_{1j} H_{2(j+1)} = -H_{2j} H_{1(j+1)} = -H_{3j}(j = 1, 3, 5) \).

Generally, there are two methods, namely transfer matrix method and assemble matrix method, for obtaining the dispersion relations in laminates \[8,23\]. Although the procedures of these two methods seem different, they are identical in principle by both satisfying traction-free boundary conditions on the outer surfaces of the laminate and continuity of interface conditions between two adjacent laminas in different manner. Both methods can calculate dispersion curves in a general laminate with an arbitrary stacking sequence.

Using Eq. (17), it may be observed that symmetric and anti-symmetric wave modes in general laminates cannot be decoupled. However, in designing the composite structures, symmetric laminates are practically used. A robust method is proposed to separate the two types of wave modes by imposing boundary conditions at both top and mid-plane surface. Traction-free boundary conditions on the top surface of the laminate are given by
\[ \left\{ \sigma_z, \tau_{yz}, \tau_{xz} \right\}_{z=h/2} = 0 \] (21)
Because of the symmetric geometry and symmetric material property of the laminate, only half of the laminate needs to be considered and then the following conditions on the stress and displacement components at the mid-plane for symmetric modes are imposed
\[ \left\{ w, \tau_{yz}, \tau_{xz} \right\}_{z=0} = 0 \] (22)
Likewise, the boundary conditions of anti-symmetric modes at the mid-plane are
\[ \left\{ u, v, \sigma_z \right\}_{z=0} = 0 \] (23)
By imposing displacement and stress continuity conditions along the interfaces of half layup of an \( N \)-layered laminate, total \( 3N \) equations are constructed if the assemble matrix method is used. Then set the determinant of the \( 3N \) equations to zero, and numerically solve the resulting transcendental equation for the dispersion relations of Lamb waves in symmetric laminates. In this paper, the transfer matrix method is adopted due to its stable nature of the numerical implementation \[8\].

3. Velocity dispersions and characteristic wave curves

The dispersion relation between \( \omega \) and \( k \) can be symbolically represented by an implicit functional form \( \mathcal{G}(\omega, k) = 0 \), or \( \mathcal{G}(\omega, k, \theta) = 0 \). It is assumed that this relation may be explicitly solved in the form of real roots of
\( \omega = \mathcal{W}(k) \), or \( \omega = \mathcal{W}(k, \theta) \). There are an infinite number of possible solutions, in general, with different functions \( \mathcal{W} \). Such solutions correspond to different wave modes. For plane waves, the phase velocity vector is defined as \( c_p = (\omega/k)(k/|k|) = (\omega/k^2)|k| \) and thus its magnitude is \( c_p = \omega/k \). A curve generated by all choices of \( k \) from the origin for \( c_p \) at a given frequency is called velocity curve. The radius vectors of velocity curves in the direction of a given \( k \) represent the admissible phase velocity dispersion of different wave modes.

Similarly a slowness (or inverse velocity) curve can be introduced by defining a slowness vector \( s = k/\omega \). The slowness curve can be simply formed from the velocity curve by geometric inversion, i.e., by the mapping through reciprocal radius. The slowness vector has the same direction as the phase velocity vector. Thus the inverse of the phase velocities can be measured from the origin to the slowness curves. Phase velocity is numerically equal to the distance traveled by a wave curve shown in Fig. 2 b. Based on Eqs. (25), it can be proved that the wave normal vector \( \mathbf{n} \) is parallel to the normal direction of the wave curve as shown in Fig. 2a. Similarly it can be shown that the wave normal vector \( \mathbf{k} \) is parallel to the normal direction of the slowness curve. In polar coordinates, the tangent vector of slowness curve \( \mathcal{W} = \mathcal{W}(k, \theta) \) at \( \mathcal{W}(k, \theta) = \omega_0 \) can be obtained by differentiating both sides of the equation with respect to \( \theta \) yields

\[
\frac{\partial \mathcal{W}}{\partial k} \frac{dk}{d\theta} + \frac{\partial \mathcal{W}}{\partial \theta} = 0
\]

In polar coordinates, the tangent vector of slowness curve \( \mathcal{W}(k, \theta) = \omega_0 \) is proportional to \( (\mathbf{\omega I}, \mathbf{k I}) \) and group velocity in polar coordinates, Eq. (25), is \( (\mathbf{\omega I}/C_0, \mathbf{k I}/C_1) \) where \( \mathbf{I} \) and \( \mathbf{I} \) are the unit vector along and perpendicular to the \( k \)-direction, respectively. It can be concluded from Eq. (28) that the group velocity vector \( \mathbf{c}_g \) is perpendicular to the tangent vector of slowness curve; that is, the group velocity vector \( \mathbf{c}_g \) is parallel to the normal direction of slowness curve as shown in Fig. 2a. Similarly it can be proved that the wave normal \( \mathbf{k} \) is parallel to the normal direction of wave curve shown in Fig. 2b.

Although group velocity expressions in Eqs. (25) and (29) are physically equivalent, they are suitable for different...
numerical implementations: Eq. (25) is conveniently employed to compute group velocity dispersions along a given wave propagation direction; while Eq. (29) is more suited to calculating wave curves at a given frequency.

To obtain the group velocity dispersion at a given wave propagation direction $h_1$, two dispersion relations $x = \mathcal{W}(k, h)$ having slightly different directions of propagation $h_1 \pm \Delta h/2$ are required. Then $\partial \mathcal{W} / \partial k$ in Eq. (25) is calculated from one of the two obtained dispersion relations, and the derivative term $\partial \mathcal{W} / \partial h$ in Eq. (25) can be approximated by finite central difference:

$$\frac{\partial \mathcal{W}}{\partial h} \approx \frac{\mathcal{W}(k)|_{h_1 + \Delta h/2} - \mathcal{W}(k)|_{h_1 - \Delta h/2}}{\Delta h}$$

For computation of a wave curve at a given frequency, a semi-exact method is developed by performing finite difference on the exact solutions of two slowness curves corresponding to two very close frequencies, $\omega_1 \approx \omega_2$. $\partial \mathcal{W} / \partial k$ in Eq. (29) can be calculated by

$$\left. \frac{\partial \mathcal{W}}{\partial k} \right|_{\omega=\omega_1} \approx \frac{\omega_2 - \omega_1}{k_2(\theta) - k_1(\theta)}$$

where $k_1$ and $k_2$ are both function of $\theta$. In addition, $dk/d\theta$ can be computed from one of the two known slowness curves.

Note that the geometric relation above is also valid for bulk (non-dispersive) waves. In addition, the polar reciprocal of the slowness curve is the wave curve (i.e., $s \cdot c_g = 1$)

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Material properties of AS4/3502 composite lamina</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$ (GPa)</td>
<td>$E_2$ (GPa)</td>
</tr>
<tr>
<td>127.6</td>
<td>11.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Properties of two AS4/3502 laminates</th>
</tr>
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<tbody>
<tr>
<td>Specimen</td>
<td>Stacking sequence</td>
</tr>
<tr>
<td>I</td>
<td>[+45/−45]s</td>
</tr>
<tr>
<td>II</td>
<td>[+45/−45/0/90]s</td>
</tr>
</tbody>
</table>

Fig. 3. Dispersion curves of Lamb waves traveling along $\theta = 30^\circ$ in the laminate $[\pm 45\degree /-45\degree]_s$: (a) $c_p$ of symmetric modes; (b) $c_g$ of symmetric modes; (c) $c_p$ of anti-symmetric modes; (d) $c_g$ of anti-symmetric modes.
and \( c_p = c_\theta \cos \theta \) [24]. However, these two relations break down for Lamb waves in lamina and laminates because of the dispersive behavior [10].

4. Numerical results

The formulation described in the previous sections is implemented by Matlab\textsuperscript{®}, because it can seamlessly combine symbolic and numeric computation. The composite material used in this study is AS4/3502 graphite/epoxy as shown in Table 1. Two laminates \([+45/–45/0/90]_s\) and \([+45/–45/0/90]_s\) are used in the tests and their dimensions are listed in Table 2. Numerical results consist of dispersion curves (phase and group velocities) and three characteristic wave curves in two different types of laminates. All figures of this section are displayed in both dimensionless (located at the bottom and left sides) and dimensional (located at the top and right sides) axes. Dimensionless frequency \( \omega h/c_T \) and dimensionless velocity \( c_p/c_T \) and \( c_g/c_T \) are employed to normalize the physical frequency and velocity, respectively. Additionally \( c_T \) defined as \( \sqrt{G_{12}/\rho} \) is the transverse (in-plane shear) wave velocity in the lamina.

A set of Figs. 3 and 4 shows the dispersive curves of Lamb waves in two types of laminates. Five symmetric and six anti-symmetric wave modes are shown in each figure. All Lamb waves have cut-off frequencies with the exception of the fundamental modes (\( A_0 \), \( S_0 \), and \( SH_0 \)). Note that the interaction of Lamb wave modes with delamination was most analyzed in the low frequency range where only the fundamental modes exist. In Fig. 3 a and b the \( SH_0 \) and \( S_0 \) modes are little dispersive in the low frequency range, below frequency \( \omega h/c_T = 0.5 \). Thus the different frequency components within the wave packet propagate at almost the same velocity and thus the wave packet retains its shape as it travels. In addition to this desired feature, less attenuation compared to \( A_0 \) waves [25] and high sensitivity to the delamination are other two reasons that have promoted interest using symmetric modes as diagnostic waves [25,26]. However it is difficult to diagnose the damage with such mode in practice because \( S_0 \) mode is relatively weak in magnitude compared to that of \( A_0 \) mode if two modes are simultaneously excited. Therefore other researchers prefer \( A_0 \) mode [16,27]. Besides larger magnitude, \( A_0 \) mode offers superior resolution than
$S_0$ and $SH_0$ modes because the wavelength of $A_0$ mode is always shorter than that of $S_0$ mode, especially at low frequency range. In the higher frequency range, Lamb wave propagation in the relatively thick symmetric angle-ply laminate $[+45/-45]_s$ has a complex behavior as shown in Fig. 3. The $SH_0$ and $S_0$ modes are highly dispersive for the group velocity dispersion in Fig. 3b. Similarly, it is evident from Fig. 4 that dispersions of symmetric modes in the quasi-isotropic layup $[+45/-45/0/90]$ is much stronger. On the contrary, the dispersion of $A_0$ mode in both laminates is weaker beyond $\omega h/c_T = 1$ as shown in Figs. 3d and Fig. 4d, and this feature is desirable for SHM.

Figs. 5 and 6 show the characteristic wave curves including velocity, slowness, and wave curves of Lamb waves propagating in composites at a given frequency. All the curves are centro-symmetric with respect to the origin because the fiber orientation of each individual lamina is invariant with the replacement of $\theta$ by $\theta + \pi$. Moreover, all characteristic wave curves vary with the frequency due to dispersive nature. In Fig. 5, all characteristic wave curves are plotted at $\omega h/c_T = 4$ with two symmetric modes ($S_0$ and $SH_0$) and three anti-symmetric modes ($A_0$, $SH_1$, and $A_1$) shown. The wavelengths of these modes at $\theta = 30^\circ$ are $\lambda_{S0} = 3.91$ mm, $\lambda_{SH1} = 8.02$ mm, $\lambda_{A1} = 24.7$ mm, $\lambda_{SH1o} = 5.98$ mm, and $\lambda_{A0} = 7.70$ mm, in the same order compared to the plate thickness but much larger than the fiber diameters and fiber spacing. Note that the characteristic wave curves can differ considerably for different frequencies due to its dispersive nature.

As shown in Fig. 5c and f, an interesting phenomenon of energy focusing, well-known for bulk waves in anisotropic solids, occurs for $SH$ wave modes as well. Slowness curves have many inflexion points. It implies that the same group velocity direction may correspond to several directions of wave propagation [28]. These particular shapes are responsible for the cusps of the associated wave curve and the cusps are material orientation dependent.

In Fig. 6, the characteristic wave curves of thin quasi-isotropic laminate $[+45/-45/0/90]$ are shown. The frequency is chosen as $\omega h/c_T = 1.78$ which is below the cut-off frequencies of $A_1$ and $S_1$ modes, thus only the fundamental modes ($A_0$, $S_0$, and $SH_0$) exist. The angular dependence of Lamb waves in the laminate $[+45/-45/0/90]$, becomes weaker because of quasi-isotropic layup. While, it can still be discerned that $A_0$ mode has maximum along $45^\circ$ (or $225^\circ$) directions because the bending dominant outer lamina is orientated in these directions. Moreover, there is no cusp in each wave curve. This fact results from the quasi-isotropic layup rather than dispersion characteristics. Since the velocity curves are approximately independent of the direction of wave propagation, the average wavelengths can be evaluated from Fig. 6a and d as $\lambda_{S0} = 11.8$ mm, $\lambda_{SH10} = 6.83$ mm, and $\lambda_{A0} = 2.88$ mm, comparable to the plate thickness but much larger than the fiber diameters and fiber spacing. It also shows that the wavelength of $A_0$ is shorter than that of $S_0$ and $SH_0$ modes.

5. Experimental results

In this section, the dispersive and anisotropic behavior of Lamb wave propagation in two laminates will be obtained from experiments, and then compared with the theoretical results presented in the previous section. A pair of lead zirconium titanate (PZT) disks (Navy Type II, PKI-502) serving as actuator is mounted on the opposite side of top and bottom surfaces of the laminate. The diameter and thickness of PZT disks are 6.4 and 1.57 mm, respectively. A setup schematic of PZT actuators is illustrated in Fig. 7, where PZT actuators are bonded by Loctite$^\text{TM}$ Extra Time Epoxy and the vertical arrows denote polarity directions and horizontal arrows indicate deformation of PZTs. A piezoelectric sensor (PAC Micro-80) is chosen to receive the response waves due to its wide band and small size (10 mm diameter). Both “induced strain” piezoelectric actuator and sensors utilize the $d_{31}$ mode to generate and receive stress waves [29]. Micro-80 is temporally bonded on the laminate surface at 10 cm away from the actuators by 2211 silicone compound vacuum grease coupler. Since the grease coupler is removable, it is convenient to measure the Lamb wave signals at different directions of wave propagation. According to the calibration certificate of Micro-80, micro-80 is sensitive in a wide frequency range, typically from 100 kHz up to 1 MHz. Note that since the piezo sensor is sensitive over the mounted area, the resulting voltage is the averaged voltage over this area. This implies that the sensor sensitivity in general will decrease for wavelengths smaller than the diameter of the piezo sensor.

Fig. 8 displays the diagram and actual photo of experimental setup. An Angilent 3220A function generator outputs a five-peaked tone burst signal. Then the signal is sent to TDS420A oscilloscope and K-I7602 amplifier as well, and the peak-to-peak voltage of the excitation signal from the amplifier is kept at 40 V. Then the response signal collected by the sensor is displayed and stored in the digital oscilloscope, whose sampling rate is set to 5 MHz and storage is set to 1000 data points for each test. Finally, a computer obtains the collected data via GPIB interface bus and runs Gabor wavelet transform to extract group velocities from arrival times of Lamb modes.

The function generator generates a transient $N$-peaked tone burst input voltage signal governed by

$$V_{in}(t) = P[H(t) - H(t - N_p/f_c)] \left(1 - \cos \frac{2\pi f_c t}{N_p} \right) \sin 2\pi f_c t$$  \hspace{1cm} (32)

where peak number $N_p = 5$, constant $P = 10.25$ is signal intensity, $f_c$ is the central frequency and $H(t)$ is Heaviside step function. It can be found through frequency domain that the frequency components of this excitation are mainly concentrated in a small range around the central frequency $f_c$, thus the dispersive effect can be significantly reduced. Furthermore, the frequency bandwidth of the input signal is proportional to $f_c/N_p$. Although frequency resolution
Fig. 5. Velocity, slowness, and wave curves of Lamb waves in the laminate [+45/−45]s at \(\text{ob/hc_T} = 4\): (a) Velocity curves of symmetric modes; (b) slowness curves of symmetric modes; (c) wave curves of symmetric modes; (d) velocity curve of anti-symmetric mode; (e) slowness curve of anti-symmetric mode; (f) wave curve of anti-symmetric mode.
Fig. 6. Velocity, slowness, and wave curves of Lamb waves in the laminate \([+45/-45/0/90]_s\) at \(\phi h/c_T = 1.78\): (a) velocity curves of symmetric modes; (b) slowness curves of symmetric modes; (c) wave curves of symmetric modes; (d) velocity curves of anti-symmetric modes; (e) slowness curves of anti-symmetric modes; (f) wave curves of anti-symmetric modes.
can be improved by increasing the peak number \( N_p \) of excitation signal, the time resolution worsens because of Heisenberg's Uncertainty Principle [30].

In the tests of group velocity dispersions, the sensor location is fixed (i.e., the direction of wave propagation is given) and \( f_c \) is varied from 50 kHz to 1 MHz in increments of 50 kHz. This directly corresponds to the time duration from 5 to 100 \( \mu \)s. For measuring wave curves for a given \( f_c \), the sensor locations are varied so that the actuator-to-sensor orientation changes from 0° (\( x \)-direction) to 90° (\( y \)-direction) with 10° increment, varying the direction of wave propagation. Note that the distance between sensor and actuator maintains 10 cm for all tests, except at frequency 50 kHz where the distance is 20 cm such that sufficient arrival time can be deduced between the excitation signal and the signal collected by the sensor.

In order to enhance the contrast of the arriving wave packets and to increase the precision regarding their arrival times, Gabor wavelet transform is introduced to process the received signals of Lamb waves. The Gabor wavelet transform has been demonstrated as a post signal processing technique for analyzing dispersive waves in SHM [27,31,32]. The theory and physical interpretation of Gabor wavelet transform are not repeated here. Kishimoto et al. [31] have shown that by using the peak of the magnitude of wavelet coefficients the arrival times of the group velocity at each local frequency can be extracted. Accordingly, the procedure of extracting arrival times by Gabor wavelet transform is listed as follows: (1) computing wavelet coefficients of the response signal in time-scale domain, (2) picking up a time slice from the magnitude of wavelet coefficients at the local frequency equal to the central frequency.
frequency of excitation, (3) storing the time instants of each peak in the time slice, which are associated with arrival times of Lamb modes. The group velocity can be easily attained by dividing the propagation length (actuator-to-sensor distance) by the arrival time.

Figs. 9 and 10 show compared results between theoretical prediction and experimental measurement of Lamb waves in the relatively thick laminate [+45/−45/0/90]s. It can be seen that the exact solutions match well with the experimental results for both symmetric and anti-symmetric modes. Even the higher modes such as S1 and A1 can be detected in the experiment. While, it is hard to distinguish SH0 and S0 modes at very low frequency range (50–150 kHz) as shown in Fig. 9a, because the difference of the arrival time between the two modes is very small (typically less than 10 μs), and the two modes have already appeared even before the end of the excitation duration. Furthermore, in Fig. 9a the exact solution of S2 modes does not agree with the experimental results. The disagreement may result from larger scattering signals from the heterogeneities at high frequency range for the lower wave modes than those from the S2 mode itself. The cusps of the SH0 and SH1 wave curves shown in Fig. 10 are observed in these measurements.

The compared results of Lamb waves in quasi-isotropic laminate [+45/−45/0/90]s are displayed in Figs. 11 and 12. For the excitation frequencies up to 1 MHz, the eight-layered thinner composite only exhibits fundamental guided waves (SH0, S0, and A0) propagating in the laminate. It can be also found that the exact solutions make good agreements with the experimental measurements for both group velocity dispersions and wave curves.

6. Conclusions

Exact solutions of Lamb waves in a lamina are established from 3-D elasticity theory, and then they are extended to a general laminate with an arbitrary layup. For symmetric laminates, a robust method is proposed to decouple the wave modes. Then the physical relations
between phase, slowness, and group velocity are discussed, and the characteristic wave curves are introduced to analyze the angular dependence of Lamb waves. Meanwhile, a semi-exact method is developed to obtain group velocity dispersions and wave curves. Then the methods are numerically implemented to show the dispersions and characteristic wave curves of Lamb waves in two types of laminates. In experiments, two piezoelectric actuators are employed to excite pure symmetric or anti-symmetric wave modes, and Gabor wavelet transform is adopted to obtain the group velocities of Lamb waves. The induced strain piezoelectric actuator/sensor can capture all the wave modes, not like interferometer only $S$ and $A$ modes can be detected [10]. Comparing the results between theoretical predictions and experimental measurements, it can be seen that the proposed methods effectively model the dispersive and anisotropic behavior of Lamb waves in laminates. It is also suggested that $A_0$ mode Lamb wave is more suitable for SHM, multiple higher-order Lamb waves can be excited from piezoelectric actuators and the measured group velocities agree well with those from 3-D elasticity theory. Future studies will apply the proposed methods together with piezo-excitation/collection technique to Lamb-wave-based SHM in composites. Due to the deficiency of intensive computation using 3-D theory, future study may include developing a new higher-order plate theory to significantly reduce computational cost and more accurately approximate the exact solutions.

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References


Fig. 12. Theoretical and experimental results of wave curves in the laminate [+45/−45/0/90], at $f = 500$ kHz ($\omega h/c_T = 1.78$): (a) wave curves of symmetric modes; (b) wave curve of anti-symmetric mode.